Method for Distributed Forward Dynamic Simulations of Constrained Mechanical Systems

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Abstract

In this paper, we propose the modular development of the direct violation correction method for the distributed computation of the forward dynamics simulations of constrained mechanical systems. We exploit the natural spatial parallelism of closed-chain linkages, initially, for the modular development of overall dynamics and subsequently, for the distributed numerical simulation of the dynamics. The technique could be implemented for various systems of connected bodies with variable number degrees of freedom such as systems with coulomb frictions. A numerical example is provided to demonstrate the effectiveness and stability of this method.

1 Introduction

The most significant conditions of simulation of mechanical systems are: stability, numerical efficiency, distributive computation and wide adaptability. Many methods have been proposed and implemented in commercial codes for the simulation of constrained mechanical systems; see, e.g. [1-4]. Most of them are based on the Lagrangian equations of motion:

$$\begin{aligned} &\dot{q} = v \\ &M(q)\dot{v} = f(q,v) - G(q)^T \lambda \\ &g(q) = 0, \end{aligned}$$

where

q is the vector of generalized coordinates,
v is the vector of generalized velocities,
M(q) is the mass matrix,
f(q,v) is the vector of external forces (other than constrain forces),
g(q) is the vector of (holonomic) constraints,
$G(q) = \dfrac{\partial g}{\partial q}$ is the constraint Jacobian matrix,
λ is the vector of Lagrange multipliers.

Such methods have two disadvantages: the first is that they are inconvenient for the simulation of constrained mechanical systems with a variable number of degrees of freedom. And the second is that the dynamics of multibody systems cannot be simulated distributively.

Our goal is to develop an object-oriented method for the distributed computation of the forward dynamics simulations of multibodies with variable number of degrees of freedom. In this paper we show the implementation of the method in the case of conservative systems with holonomic constraints but it could be also extended for the simulation the non-conservative systems with different types of constraints.

2 Translation

Step 1: Translation of subsystem

Consider the subsystem of n connected bodies. Let first *m* bodies be connected with external joints with the complete system.

Let x_i denotes the 6-lengths vector of global coordinates and Euler angles of body *i*. Let *X* denote the *n*-lengths vector consists 6-lengths x_i:

$$X = (x_1^T,...,x_n^T)^T.$$

Let X_e denote the *m*-lengths subvector of vector *X* consisting of coordinates and angles of bodies which are connected with external joints. Let X_i denote the *(n-m)*-length subvector of vector *X* consisting of coordinates and angles of bodies which are not connected with external joints. Obviously:

$$X = (X_e, X_i)^T.$$

Let *k* denote the number of internal constraints. Thus, differentiating the internal constraints

$$g = (g_1(X),...g_k(X))^T = (0,...,0)^T$$

once, we obtain the constraint equation on the velocity level:

$$0 = \dot{g} = \begin{pmatrix} \frac{\partial g}{\partial X_e} & \frac{\partial g}{\partial X_i} \end{pmatrix} \begin{pmatrix} \dot{X}_e \\ \dot{X}_i \end{pmatrix} = (G_e \quad G_i) \begin{pmatrix} \dot{X}_e \\ \dot{X}_i \end{pmatrix} = G\dot{X}.$$

Let *Q* denote the Lagrange forces acting in external constraints. Let Y_i denote Lagrangian forces assotiated with external forces and torques acting on *i*-th body. The Lagrangian equations:

$$\frac{d}{dt}\frac{\partial T_i}{\partial \dot{x}_i} - \frac{\partial T_i}{\partial x_i} = \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i} + Q_i + Y_i \qquad i = 1...m$$
$$\frac{d}{dt}\frac{\partial T_i}{\partial \dot{x}_i} - \frac{\partial T_i}{\partial x_i} = \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i} + Y_i \qquad i = m+1...n$$

can be written in matrix form:

$$M_e \ddot{X}_e + C_e = G_e^T \lambda + Q$$
$$M_i \ddot{X}_i + C_i = G_i^T \lambda.$$

On each time step let's obtain matrix *H* from matrix *G* by excluding the dependent rows. Obviously, we can represent matrix *G* as:

$$G = (G_e \quad G_i) = T \cdot H = T(H_e \quad H_i) = T \cdot \begin{pmatrix} \frac{\partial a}{\partial X_e} & \frac{\partial a}{\partial X_i} \end{pmatrix},$$

where the vector *a(x)* is obtained from the vector g(x) by excluding the same rows as in transformation from matrix *G* to matrix *A*. Obviously, the Lagrangian equations can be rewritten:

$$M_e\ddot{X}_e + C_e = H_e^T\Lambda + Q \qquad (1)$$
$$M_i\ddot{X}_i + C_i = H_i^T\Lambda,$$

where

$$\Lambda = T^T\lambda.$$

Differentiating twice the equations of internal constraints:

$$a(X) = 0,$$

we get the constraint equations on acceleration level:

$$0 = \begin{pmatrix} H_e & H_i \end{pmatrix}\begin{pmatrix} \ddot{X}_e \\ \ddot{X}_i \end{pmatrix} + \begin{pmatrix} \dot{H}_e & \dot{H}_i \end{pmatrix}\begin{pmatrix} \dot{X}_e \\ \dot{X}_i \end{pmatrix} = H_e\ddot{X}_e + H_i\ddot{X}_i + U.$$

Substituting the accelerations from the Lagrangian equations, we obtain the dependency of internal forces on forces in external links:

$$\Lambda = SQ + V, \qquad (2)$$

where

$$S = -P^{-1}H_eM_e^{-1}$$
$$V = P^{-1}(H_eM_e^{-1}C_e + H_iM_i^{-1}C_i - U)$$
$$P = (H_eM_e^{-1}H_e^T + H_iM_i^{-1}H_i^T).$$

On each time step matrix *P* can be inverted, because

$$P = (H_eM_e^{-1}H_e^T + H_iM_i^{-1}H_i^T) = H\begin{pmatrix} M_e & 0 \\ 0 & M_i \end{pmatrix}^{-1}H^T,$$

where $M_i, M_e > 0$ and $H = (H_i, H_e)$ is a matrix with independent rows.

Substituting the internal forces in the equations of motion, we obtain:

$$\ddot{X}_e = DQ + R, \qquad (3)$$

where

$$D = M_e^{-1}H_e^TS + M_e^{-1}$$
$$R = M_e^{-1}H_e^TV - M_e^{-1}C_e.$$

Step2: Hierarchy

Consider the subsystem *S* consists of *N* subsystems $S_1, S_2, \ldots, S_N$. Let $X_e^{(i)}$ denote the vector of coordinates and angles of bodies which are connected with joints external to S_i. We could separate $X_e^{(i)}$ on two subvectors: $X_e^{(i)} = (X_{ein}^{(i)}, X_{ext}^{(i)})$, where $X_{ext}^{(i)}$ are coordinates and angles of vectors of bodies which are included in constraints external to global subsystem S.

Let us denote the vector of coordinates of bodies with constraints between subsystems $X_{ein}= (X_{ein}^{(i)},..,X_{ein}^{(N)})$ and the vector of coordinates of bodies without them $X_{ext}= (X_{ext}^{(i)},..,X_{ext}^{(N)})$.

Differentiating the constraints between subsystems $S_1,S_2,\ldots,S_N$

$$g=(g_1(X_{ein},X_{ext}),...,g_k(X_{ein},X_{ext}))^T=(0,...,0)^T.$$

once, we obtain the constraint equatuon on the velocity level:

$$0=(G_{ext} \quad G_{ein})\begin{pmatrix}\dot{X}_{ext}\\ \dot{X}_{ein}\end{pmatrix}=\begin{pmatrix}\dfrac{\partial g}{\partial X_{ein}} & \dfrac{\partial g}{\partial X_{ext}}\end{pmatrix}\begin{pmatrix}\dot{X}_{ext}\\ \dot{X}_{ein}\end{pmatrix}=G\dot{X}_e.$$

A further differentiation with respect to time results in the constraint equation on acceleration level:

$$0=\left(G_{ext} \quad G_{ein}\right)\begin{pmatrix}\ddot{X}_{ext}\\ \ddot{X}_{ein}\end{pmatrix}+\left(\dot{G}_{ext} \quad \dot{G}_{ein}\right)\begin{pmatrix}\dot{X}_{ext}\\ \dot{X}_{ein}\end{pmatrix}=G_{ext}\ddot{X}_{ext}+G_{ein}\ddot{X}_{ein}+U.$$

On each time step let's exclude dependent rows from matrix G and exclude the same rows from the vector g(x).

Let λ denote Lagrange multipliers associated with the constraints between subsystems $S_1,S_2,\ldots,S_N$ and Q' denote forces acting in links external to subsystem S. From previous hierarchy level we get matrices $D^{(i)}$ and $R^{(i)}$. We can write equations of accelerations

$$\ddot{X}_e^{(i)}=D^{(i)}Q^{(i)}+R^{(i)} \qquad i=1...N$$

in the matrix form:

$$\begin{aligned}\ddot{X}_{ext}&=d_{ext}G_{ext}{}^T\lambda+d'_{ext}Q'+r_{ext}\\ \ddot{X}_{ein}&=d_{ein}G_{ein}{}^T\lambda+r_{ein}.\end{aligned} \tag{4}$$

Substituting the accelerations in constraint equation on acceleration level, we obtain the dependency of internal forces λ on forces in external links Q':

$$\lambda=SQ'+V, \tag{5}$$

where

$$\begin{aligned}S&=-(G_{ext}d_{ext}G_{ext}{}^T+G_{ein}d_{ein}G_{ein}{}^T)^{-1}G_{ext}d'_{ext}\\ V&=-(G_{ext}d_{ext}G_{ext}{}^T+G_{ein}d_{ein}G_{ein}{}^T)^{-1}(G_{ext}r_{ext}+G_{ein}r_{ein}+U).\end{aligned}$$

Substituting internal forces in the equations of motion, we obtain:

$$\ddot{X}_{ext}=DQ'+R, \tag{6}$$

where

$$\begin{aligned}D&=d_{ext}G_{ext}{}^TS+d'_{ext}\\ R&=d_{ext}G_{ext}{}^TV+r_{ext}.\end{aligned}$$

Step3: The complete model description

We should iteratively perform the second steep until the subsystem includes all bodies.

If our multibody system S is connected with the ground, whose motion is known: $X_0=X_0(t)$, then on the last step we should include the ground in the model.

Let X_{ein} denote vector with coordinates and angles of bodies which are connected with the ground. Differentiating the constraints between system *S* and the ground

$$g=(g_1(X_0,X_{ein}),\dots,g_k(X_0,X_{ein}))^T=(0,\dots,0)^T.$$

once, we obtain the constraint equation on the velocity level:

$$0=(G_0 \quad G_{ein})\begin{pmatrix}\dot X_0\\ \dot X_{ein}\end{pmatrix}=\begin{pmatrix}\frac{\partial g}{\partial X_0} & \frac{\partial g}{\partial X_{ext}}\end{pmatrix}\begin{pmatrix}\dot X_0\\ \dot X_{ein}\end{pmatrix}=G\dot X_e.$$

A further differentiation respect to time results in the constraint equation on acceleration level:

$$0=\begin{pmatrix}G_0 & G_{ein}\end{pmatrix}\begin{pmatrix}\ddot X_0\\ \ddot X_{ein}\end{pmatrix}+\begin{pmatrix}\dot G_{ext} & \dot G_{ein}\end{pmatrix}\begin{pmatrix}\dot X_0\\ \dot X_{ein}\end{pmatrix}=G_0\ddot X_0+G_{ein}\ddot X_{ein}+U.$$

On each time step let's exclude dependent rows from matrix *G* and exclude the same rows from the vector g(x).

Let λ denotes Lagrange multipliers associated with constraints between the system *S* and the ground.

The equations of accelerations are:

$$\ddot X_{ein}=DG_{ein}{}^T\lambda+R$$
$$\ddot X_0=\ddot X_0(t),$$

where matrices D and R are known from previous hierarchy level.

Substituting the accelerations in the constraint equation on acceleration level, we obtain the dependency of Lagrange multipliers on ground acceleration:

$$\lambda=-(G_{ein}SG_{ein}{}^T)^{-1}(G_0\ddot X_0+G_{ein}R_{ein}+U). \qquad (7)$$

Step4: Generation of constraint Jacobian matrix G

In our method we use the stabilization technique, proposed by Ascher [5], based on the inversion of the constraint Jacobian matrix *G* of the complete system. Obviously due to generalized coordinates instead of global coordinates and Euler angles we can significantly reduce the dimension of the matrix *G*. For example the dimension of *G* of a simple 2D loop with *n* revolute joins expressed using generalized constraints is *(n, 2)* vs. *(3n, 2n)* dimension of *G* expressed using global coordinates and Euler angles.

The main disadvantage of equations of motion expressed in generalized coordinates is that they could not be solved distributively. It happends because some of generalized coordinates could be included in Lagrangian equations of motion of all bodies. That is why we propose to translate the model using common coordinates but to simulate and to stabilize it using generalized coordinates.

Let q denote the vector of generalized coordinates of the complete system. Let v denote the vector of generalized velocities. We need to choose such set of generalized coordinates that it would be possible to obtain three dependencies in the symbolical form:

$$\begin{aligned} X &= X(q) \\ \dot{X} &= \dot{X}(q,v) \\ \ddot{q} &= \ddot{q}(X,\dot{X},\ddot{X}), \end{aligned} \tag{8}$$

where X is the vector of global coordinates and Euler angles.

The first two equations are dependency of global coordinates and velocities. The third is the backward dependency of generalized coordinates on acceleration level.

Using this equation we obtain the constraint Jacobian matrix G expressed in generalized coordinates:

$$G(q) = \frac{\partial g}{\partial q}.$$

3 Simulation

The simulation steps in each time step are shown in Fig. 1.

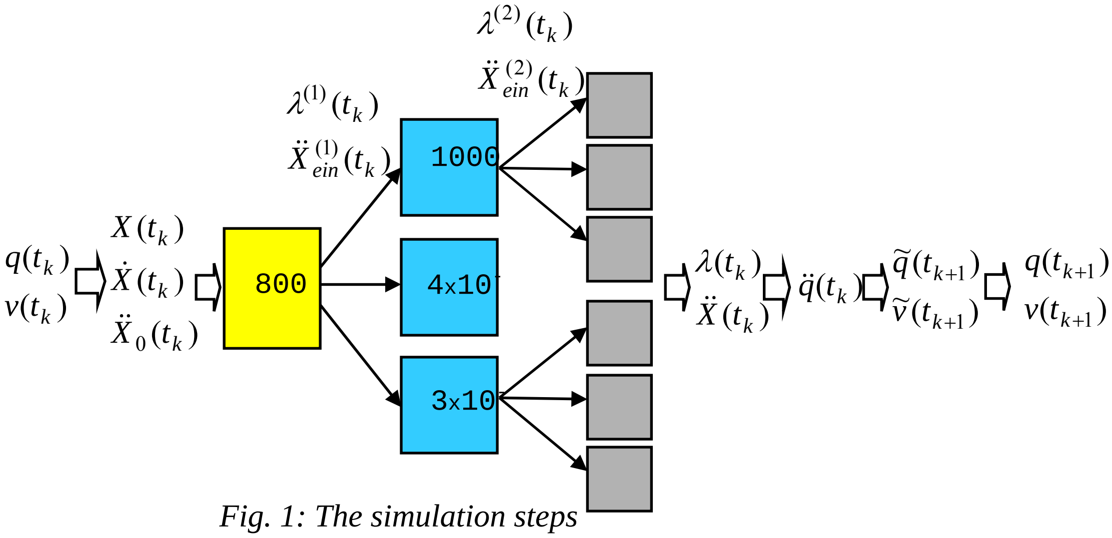

Fig. 1: The simulation steps

Step1: Using first two equations of (8) we obtain the global coordinates $X(t_k)$ and velocities $\dot{X}(t_k)$.

Step2: Substituting the global coordinates, global velocities and accelerations of the ground in equation (7), we obtain the Lagrange multipliers associated with the subsystems of the first level hierarchy $\lambda^{(1)}(t_k)$. From the equation of motion we obtain the accelerations $\ddot{X}^{(1)}_{ein}(t_k)$.

Step3: Substituting forces acting in external links in the equations of motion of subsystem we obtain the Lagrange multipliers and accelerations of links on the next hierarchy level.

Step4: Iteratively repeating step 3 we obtain all Lagrange multipliers $\lambda(t_k)$ and the vector of all accelerations $\ddot{X}(t_k)$.

Step5: From the third equation of (8) we obtain the vectors of all generalized accelerations $\ddot{q}(t_k)$.

Step6: Using the ODE integration scheme (e.g. Runge-Kutta) we calculate the vector of generalized coordinates and velocities on the next time step $((\tilde{q}(t_{k+1}),\tilde{v}(t_{k+1}))^T$.

Step7: We use stabilization technique, proposed by Ascher [5]. Let $z(q,v)$ denote the stabilization function:

$$z(q,v) = G^T (GG^T)^{-1} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} g \\ Gv \end{pmatrix}.$$

Then the vector of stabilized generalized coordinates and velocities $(q(t_{k+1}),v(t_{k+1}))^T$ could be obtained from $((\tilde{q}(t_{k+1}),\tilde{v}(t_{k+1}))^T$ by the double step:

$$\begin{pmatrix} \hat{q}(t_{k+1}) \\ \hat{v}(t_{k+1}) \end{pmatrix} = \begin{pmatrix} \tilde{q}(t_{k+1}) \\ \tilde{v}(t_{k+1}) \end{pmatrix} - z(\tilde{q}(t_{k+1}),\tilde{v}(t_{k+1}))$$

$$\begin{pmatrix} q(t_{k+1}) \\ v(t_{k+1}) \end{pmatrix} = \begin{pmatrix} \hat{q}(t_{k+1}) \\ \hat{v}(t_{k+1}) \end{pmatrix} - z(\hat{q}(t_{k+1}),\hat{v}(t_{k+1})).$$

4 Example

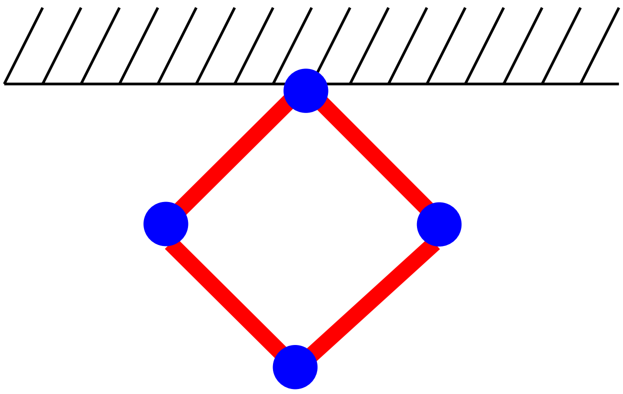

Fig. 2: Closed-loop system of 4 connected bars with revolute joints

We have performed a number of calculations for the problem of 4 connected bars with revolute joints shown in Fig. 2. The length of all bars is equal to one meter, the mass is equal to one kg.

Obviously the system has four degrees of freedom: the Euler angles $\varphi_1, \varphi_2, \varphi_3, \varphi_4$.

Let's simulate motion of the system under the action of the gravitational force, when time changes from null to 10 seconds. Simulation was performed with Runge-Kutta method of the second order with a fixed time step equal to 0.01s.

Initial conditions of the system are:

$$\varphi_1(0) = -\pi/4, \varphi_2(0) = \pi/4, \ \varphi_3(0) = 3\pi/4, \varphi_4(0) = 5\pi/4$$

$$\dot{\varphi}_1(0) = \dot{\varphi}_2(0) = \dot{\varphi}_3(0) = \dot{\varphi}_4(0) = 0$$

The motion of the system will be the oscillations shown in Fig. 3.

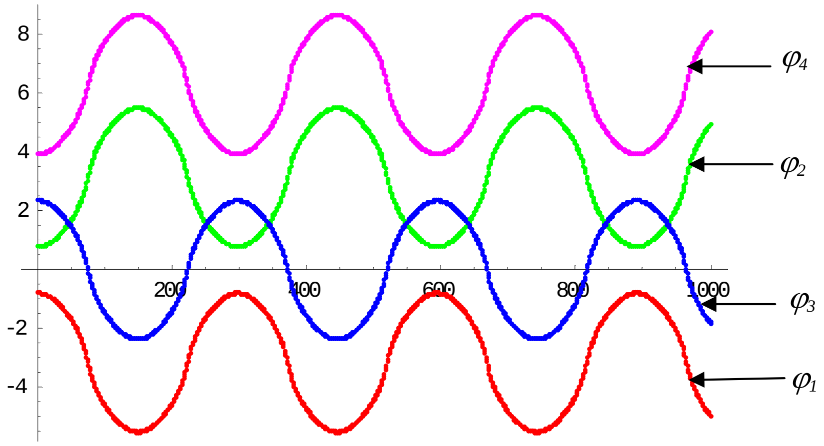

Fig. 3: The changes of Euler angles

The diagram shown in Fig. 4 illustrates the drift of the model (maximal error in the joints). Obviously our stabilization technique is highly effective and limits the drift to 5×10^{-15}m.

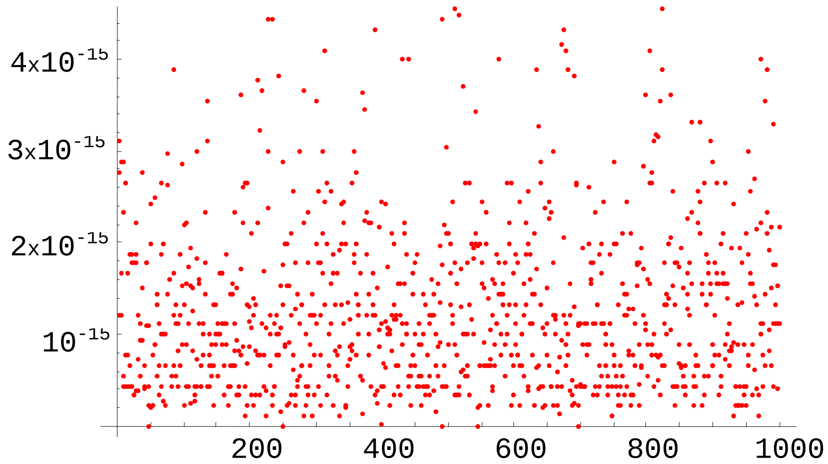

Fig. 4: The drift of the model

5 Summary

In this paper, we propose the modular development of the direct violation correction method for the distributed computation of the forward dynamics simulations of constrained mechanical systems.

In our method we widely combine the calculation using global coordinates and Euler angles and calculation using generalized coordinates. Calculation of accelerations and internal forces are done in global coordinates and Euler angles that allow us to distribute this process. For the stabilization we use generalized coordinates that makes the method numerically efficient.

The example shows that stabilization technique is highly effective and the drift of the system is limited for a long period of time.

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